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LETTER TO THE EDITOR

Spiral self-avoiding walks on a triangular lattice: end-to-end distance

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Abstract. An exact solution is obtained for the root-mean-square end-to-end distance R_N of spiral self-avoiding walks with N steps on a triangular lattice. For $N\to\infty$, R_N increases as $(2\pi)^{-1}(6N)^{1/2}\log N$.

Spiral self-avoiding walks (SSAWS) on a square lattice were first considered by Privman (1983). It is shown by Blöte and Hilhorst (1984) that, for $N\to\infty$ the number of N-step spiral walks increases as

$$S_N\sim 2^{-2}3^{-5/4}\pi N^{-7/4}\exp[2\pi(N/3)^{1/2}], \tag{1}$$

and their root-mean-square end-to-end distance behaves as

$$R_N\sim(2\pi)^{-1}(3N)^{1/2}\log N. \tag{2}$$

Similar results for S_N are derived independently by Guttmann and Wormald (1984), Joyce (1984), and Guttmann and Hirschhorn (1984). Recently one of us (Lin 1985) considered SSAWS on a triangular lattice and it is shown that, for $N\to\infty$, S_N increases as

$$S_N\sim 2^{1/4}3^{-7/4}\pi N^{-5/4}\exp[\pi(2N/3)^{1/2}]. \tag{3}$$

In a recent letter, Joyce and Brak (1985) have obtained the complete asymptotic expansion for S_N. In this letter we report an exact result for the end-to-end distance of SSAWS on a triangular lattice. We shall refer to Lin's paper as I and follow his notations unless specified otherwise.

Consider first the simpler problem of the subclass of SSAWS which only spirals outward (see figure 1 of I). The generating function is

$$\begin{aligned} G^*(z) &= \sum_{N=1}^{\infty} S_N^* z^N \\ &= \sum_{L=0}^{\infty}\sum_{0<x_1\ldots<x_L}\sum_{0<X_{L+1}} z^{x_1+\ldots+x_{L+1}} \\ &= z(1-z)^{-1}\sum_{L=0}^{\infty} g_L(z) \end{aligned} \tag{4}$$

where

$$g_0(z)\equiv 1$$

and

$$g_L(z)=\prod_{l=1}^{L} z^l(1-z^l)^{-1}, \qquad L=1,2,\ldots. \tag{5}$$

Following exactly the same procedure as Blöte and Hilhorst (1984), it can be shown by the method of steepest descent that for $z=1-\varepsilon$ and $\varepsilon\to 0^+$,

$$G^*(z)\sim 2^{-1/2}\varepsilon^{-1}\exp(\pi^2/12\varepsilon). \tag{6}$$

For $\varepsilon\to 0$, the average chain length is

$$N^*(z)=\mathrm{d}\log G^*(z)/\mathrm{d}\log z\sim \pi^2/12\varepsilon^2 \tag{7}$$

and the relative fluctuations satisfy

$$\Delta N^*/N^*\sim (N^*)^{-1/4}. \tag{8}$$

The squared end-to-end distance is

$$r_L^2=\tfrac{1}{2}[(A_L-B_L)^2+(B_L-C_L)^2+(C_L-A_L)^2] \tag{9}$$

where

$$A_L=x_1+x_4+\ldots=\sum_{k=0}^{[L/3]} x_{3k+1},$$

$$B_L=x_2+x_5+\ldots=\sum_{k=0}^{[(L-1)/3]} x_{3k+2},$$

$$C_L=x_3+x_6+\ldots=\sum_{k=1}^{([L+1)/3]} x_{3k},$$

and $[n]$ denotes the integer part of the real number n. When the expression of r^2 is inserted inside the summations of (4), we have (for $\varepsilon\to 0$)

$$F(z)\equiv\sum_{N=1}^{\infty} S_N^*(R_N^*)^2z^N\sim (R_{N^*}^*)^2\sum S_N^*z^N \tag{10}$$

where R_N^* is the root-mean-square end-to-end distance.

We define $m_1=x_1$, $m_k=x_k-x_{k-1}(k>1)$ and write

$$2r_L^2=\left(\sum_{k=0} m_{L-3k}\right)^2+\left(x_{L+1}-\sum_{k=0} m_{L-1-3k}\right)^2+\left(x_{L+1}-\sum_{k=0}(m_{L-3k}+m_{L-1-3k})\right)^2. \tag{11}$$

Using the identities

$$\sum_{m=1}^{\infty} mz^m=(1-z)^{-1}\sum_{m=1}^{\infty} z^m, \tag{12}$$

$$\sum_{m=1}^{\infty} m^2z^m=(1+z)(1-z)^{-2}\sum_{m=1}^{\infty} z^m, \tag{13}$$

we have

$$F(z)=z(1-z)^{-1}\sum_{L} R_L(z)g_L(z) \tag{14}$$

where R_L is given by

$$R_L(z)=\frac{1}{2}\left(\sum_k (1-z^{3k+1})^{-1}\right)^2+\frac{1}{2}\left((1-z)^{-1}-\sum_{k'} (1-z^{3k'+2})^{-1}\right)^2$$
$$+\frac{1}{2}\left((1-z)^{-1}-\sum_k (1-z^{3k+1})^{-1}-\sum_{k'} (1-z^{3k'+2})^{-1}\right)^2$$
$$+z(1-z)^{-2}+\sum_k z^{3k+1}(1-z^{3k+1})^{-2}+\sum_{k'} z^{3k'+2}(1-z^{3k'+2})^{-2},$$
$$\sum_k=\sum_{k=0}^{[(L-1)/3]} \qquad \sum_{k'}=\sum_{k'=0}^{[(L-2)/3]}. \tag{15}$$

It has been pointed out by Blöte and Hilhorst (1984) that at fixed z the quantity $g_L(z)$ takes maximum value for $L=L_0(z)\sim\log 2/\varepsilon$. Consequently we have

$$F(z)\sim R_{L_0}(z)G^*(z). \tag{16}$$

To calculate the leading term in R_{L_0} for $\varepsilon\to 0$, we replace the summations in (15) by integrals and it is elementary to show that

$$R_{L_0}(z)\sim(\log\varepsilon)^2/3\varepsilon^2. \tag{17}$$

It follows from (7), (10) and (16) that for $N\to\infty$

$$R_N^*\sim\pi^{-1}N^{1/2}\log N. \tag{18}$$

We now consider the proglem of all SSAWs. It follows from I that, for $z=1-\varepsilon$ and $\varepsilon\to 0$, the generating function behaves like

$$G(z)\sim 2G_6(z)-G_1(z)-G_2(z) \tag{19}$$

and

$$2G_1\sim 6G_2\sim G_6\sim\varepsilon\exp(\pi^2/6\varepsilon). \tag{20}$$

In analogy to (7), we have

$$N(z)\sim\pi^2/6\varepsilon^2. \tag{21}$$

Consider $G_1(z)$ first (see figure 3 of I). We deote the segment lengths by the integers

$$x_1, x_2, \ldots, x_{L-1}, x_L+y_{L'-1}, y_{L'-1}, \ldots, y_1, \tag{22}$$

where

$$0<x_1<\ldots<x_L, \qquad 0<y_1<\ldots<y_{L'-1}.$$

The length of the longest segment is defined to be $x_L+y_{L'-1}$. We have

$$G_1(z)=\sum_{L,L'=1}^{\infty}\sum_{x_1,\ldots,x_L}\sum_{y_1,\ldots,y_{L'-1}} z^{x_1+\ldots+x_L+2y_{L'-1}+y_{L'-2}+\ldots+y_1}$$
$$=z^{-1}(1-z)\sum_{L,L'} g_L(z)g_{L'}(z)$$
$$=z^{-1}(1-z)\left(\sum_{L=1}^{\infty} g_L(z)\right)^2. \tag{23}$$

The squared end-to-end distance is

$$(r_{L,L'})^2 = \tfrac{1}{2}[(A-B)^2 + (B-C)^2 + (C-A)^2] \tag{24}$$

where

$$A = \sum_{k=0} x_{L-1-3k} + \sum_{k'=0} y_{L'-2-3k'}$$
$$B = \sum_{k=0} x_{L-2-3k} + \sum_{k'=0} y_{L'-1-3k'}$$
$$C = \sum_{k=0} x_{L-3k} + y_{L'-1} + \sum_{k'=0} y_{L'-3-3k'}. \tag{25}$$

We define $m_i = x_i - x_{i-1}$ and $n_j = y_j - y_{j-1}$ ($m_1 = x_1$ and $n_1 = y_1$) where m_i and n_j are independent integer variables. We have

$$A - B = \sum_{k=0} m_{L-1-3k} - \sum_{k'} n_{L'-1-3k'}$$
$$C - B = \sum_{k=0} (m_{L-3k} + m_{L-1-3k}) + \sum_{k'=0} n_{L'-3-3k'} \tag{26}$$
$$C - A = \sum_{k=0} m_{L-3k} + \sum_{k'=0} (n_{L'-3-3k'} + n_{L'-1-3k'}).$$

When the expression of r^2 is inserted inside the summations of (23), we have for $\varepsilon \to 0$

$$(R_N)^2 \sim (\log \varepsilon)^2/\varepsilon^2 \tag{27}$$

where $N(\varepsilon)$ is the average chain length given by (21). Considerations of $G_2(z)$ and $G_6(z)$ yield identical results. Therefore finally we have

$$R_N \sim (2\pi)^{-1}(6N)^{1/2} \log N \tag{28}$$

for $N \to \infty$.

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